Theory and Computation for Bilinear Quadratures

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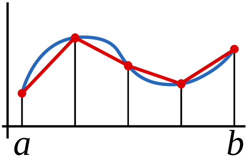
Classical and bilinear quadratures

Classical quadrature (approximates linear functionals):

$$\int_\Omega f(x)\, d\mu = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix} \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$



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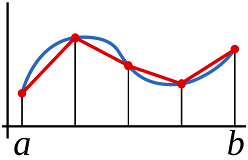

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Bilinear quadrature (approximates bilinear forms):

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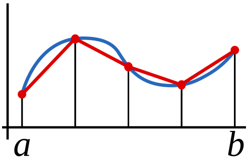

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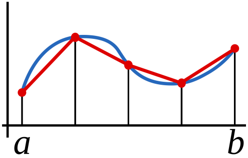

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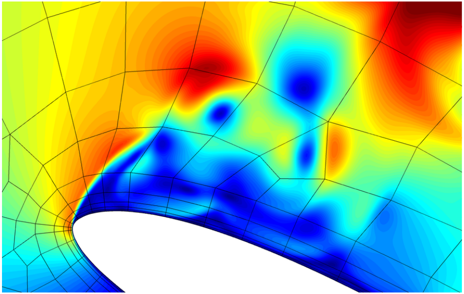
- Useful for finite element/Galerkin method.
- Previous work: Boland and Duris (1972), McGrath (1979), Gribble (1980), Chen (2012)

Advantages

- f's and g's can belong to different spaces.
- Can do any continuous bilinear form, e.g.

$$\langle f, g\rangle_{H^1} = \int_\Omega fg + Du \cdot Dv, \quad \langle f, g\rangle = \int_\Omega fg + \int_{\partial\Omega} fg.$$

  Very useful in Galerkin methods!



- The **right method** for computing an orthogonal projection onto a fixed function space.

Image courtesy of P.-O. Persson

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- Pick orthonormal basis $\{f_1, \ldots, f_k\}$ for X_0. Solve for $W = (w_{ij})$ and $\mathbf{x} = \{x_i\}$ such that:

$$F(\mathbf{x})^T W F(\mathbf{x}) = I_k$$

where

$$F(\mathbf{x}) := \begin{bmatrix} f_1(x_1) & \ldots & f_k(x_1) \\ \vdots & & \vdots \\ f_1(x_n) & \ldots & f_k(x_n) \end{bmatrix}$$

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- Approximate coefficients of $\mathrm{Proj}_{X_0}(\phi)$ using matvec $F(\mathbf{x})^T W \phi(\mathbf{x})$

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- Don't have access to derivatives. Use a quasi-Newton method (e.g. BFGS).
- Need many initial guesses to get close to the minimum.

Connection with familiar classical quadratures

Theorem

Given:

- *n-point bilinear quadrature on an interval,*
- *exact for L^2 inner products on $\mathbb{P}_{n-1} \times \mathbb{P}_{n-1}$,*
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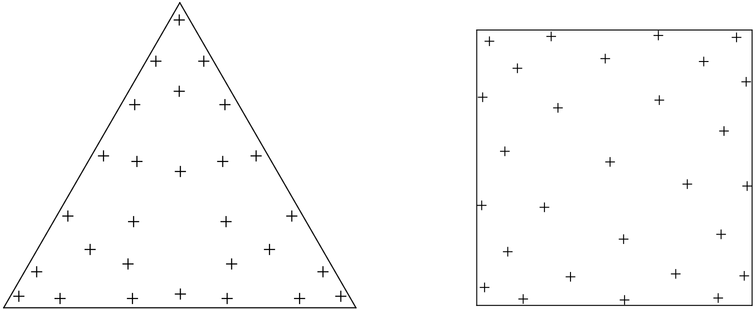
Given:

- *n-point bilinear quadrature on $[0, 2\pi]$, n odd,*
- *exact for L^2 inner products on $T_{(n-1)/2} \times T_{(n-1)/2}$,*
- *minimized over* $\operatorname{span}\{\sin \frac{n+1}{2}x, \cos \frac{n+1}{2}x\}$,

then $W = \operatorname{diag}(\frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{2\pi}{n}, \frac{\pi}{n})$ and $\mathbf{x}$ are uniformly spaced points on $[0, 2\pi]$!

Two numerical results

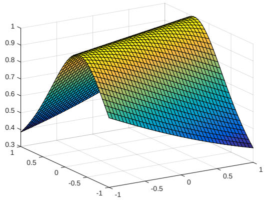


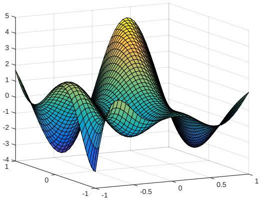
The nodes of 28-point bilinear quadratures on $\mathbb{P}_6 \times \mathbb{P}_6$, minimized against $\mathbb{P}_7$, for both the triangle and the square.
Notice symmetry!

Accuracy

- Compare with two very good 28-point classical quadratures on triangles.
- Project onto $\mathbb{P}_6$.
- Calculate relative 2-norm error in projection coefficients for random functions from four different spaces.

	$\mathbb{P}_5$	$\mathbb{P}_6$	Cauchy	Trig
Dunavant (1985)	9.38e-14	3.97e-01	4.96e-05	9.12e-03
Xiao et al. (2010)	3.29e-15	2.73e-01	1.91e-05	4.76e-03
Bilinear	3.92e-15	3.99e-15	6.70e-06	1.71e-03





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Disadvantages

- A bilinear quadrature does less than a general-purpose classical quadrature.
- Expensive to construct; optimization is slow.