Going beyond HSS: Investigating the structure of impedance operators for high frequency Helmholtz problems

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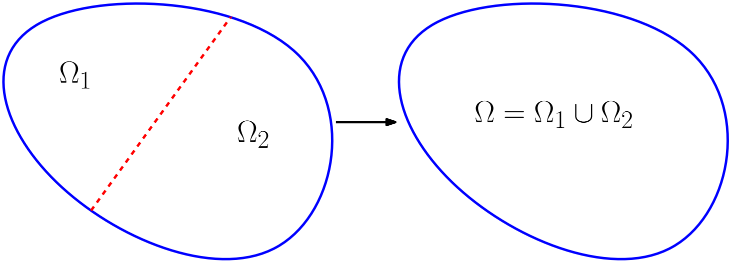
Hierarchical elliptic solvers

- Analysis-based variant of nested dissection direct solvers.
- ItI: Barnett, Gillman, Martinsson (2013), RtR: de Hoop, Liu, Xia
- Solve and merge local elliptic problems

$$A_i u = f_i \text{ on } \Omega_i$$



- Complexity: $O(n^{3/2})$ in 2D, $O(n^2)$ in 3D.

Boundary operators

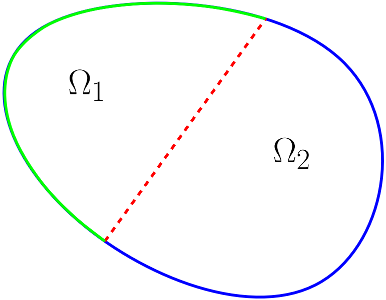
Robin-to-Robin operator (RtR) R maps one boundary condition to another:

$$\alpha u + \beta \partial_n u \mapsto \sigma u + \tau \partial_n u.$$

Impedance: $\beta = \tau = 1$, $\alpha = -i\eta$, $\sigma = i\eta$

$$R_1 \begin{bmatrix} g_1 \\ \tilde{g}_1 \end{bmatrix} = \begin{bmatrix} h_1 \\ \tilde{h}_1 \end{bmatrix},$$

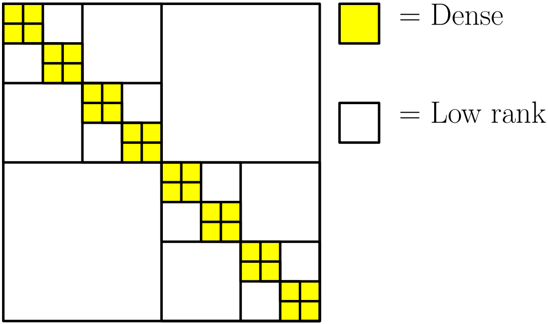
$$R_2 \begin{bmatrix} g_2 \\ \tilde{g}_2 \end{bmatrix} = \begin{bmatrix} h_2 \\ \tilde{h}_2 \end{bmatrix}$$



Construct $R \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

HSS compression

- Descended from multipole methods (Greengard, Rokhlin)
- HSS matrix compression: Chandrasekaran, Gu, Li, Pals, Xia, and others
- Accelerate elliptic direct solvers.



HSS Compression

$G(x, y)$ has ϵ-rank r on $X \times Y$ if there are functions f_i, g_i such that

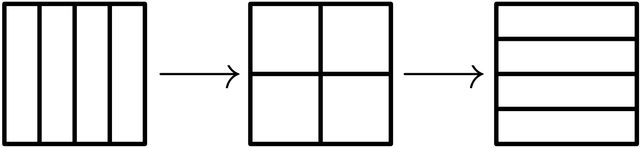
$$\left| G(x, y) - \sum_{i=1}^{r} f_i(x) g_i(y) \right| \leq \epsilon.$$

for all $x \in X, y \in Y$.

- Low frequency: r grows slowly (if at all!) as X, Y grow
- High frequency: r grows with sizes of X, Y.
- Low-rank block $\Leftrightarrow$ too many points

Butterfly compression

- Butterfly algorithm (Michielssen, Rokhlin, O'Neil, Candes, Ying, Demanet, others); connections to FFT



- Interactions with $\rho(X)\rho(Y) = O(1/k)$ are low-rank.
- Oscillatory matrix with frequency k can be applied in $O(r^2 k^d \log k)$ flops.
- Multi-level butterfly: treat off-diagonal blocks as butterfly compressible

Butterfly compression

Oscillatory integral kernel

$$G(x, y) = \exp(ik\phi(x, y)),$$

where ϕ is analytic. Expand about (x_0, y_0):

$$\begin{aligned} G(x, y) = \exp(ik\phi(x_0, y_0)) \exp(ik(\phi(x, y_0) - \phi(x_0, y_0))) \cdot \ldots \\ \cdot \exp(ikR(x, y)) \exp(ik(\phi(x_0, y) - \phi(x_0, y_0))), \end{aligned}$$

with $R(x, y)$ the 2nd-order Taylor remainder. For regions X, Y with centers x_0, y_0:

$$|R(x, y)| \leq \|\partial_x \partial_y \phi\|_\infty \rho(X)\rho(Y).$$

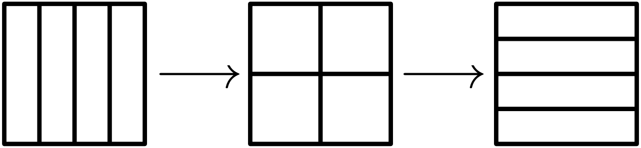
Butterfly compression

- $R(x, y) = O(1/k) \Rightarrow \exp(ikR(x, y))$ expands as

$$\exp(ikR(x, y)) \approx \sum_{j=1}^{r_1} \frac{i^j}{j!} k^j R(x, y)^j$$

- $\phi(x, y)$ analytic $\Rightarrow R(x, y)$ analytic
- Insert Taylor expansion of $R(x, y)$.

Butterfly compression



- Step 1: Compress interaction of a few sources with many targets
- Step 2: Combine sources and split targets
- Step 3: Determine interaction of all sources on a few targets

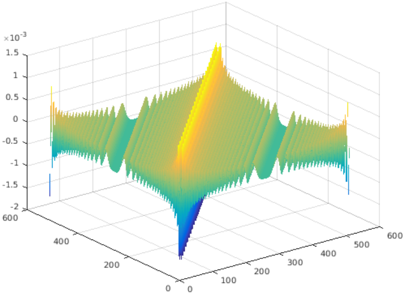
Can view as a factorization:

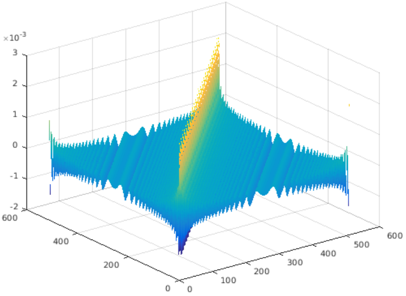
$$A = PB_1 \dots B_L Q$$

Helmholtz single-layer potential

2D Helmholtz single-layer potential on S^1:

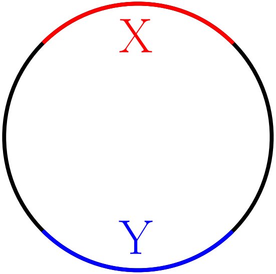
$$\mathcal{S}u(x) = \frac{i}{4}\int_{S^1} H_0^{(1)}(|x-y|)u(y)ds(y)$$

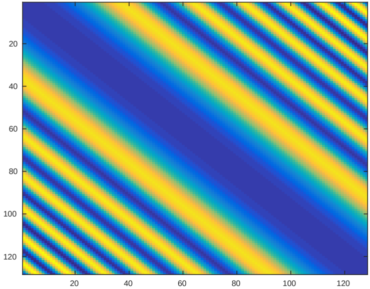




Helmholtz single-layer potential

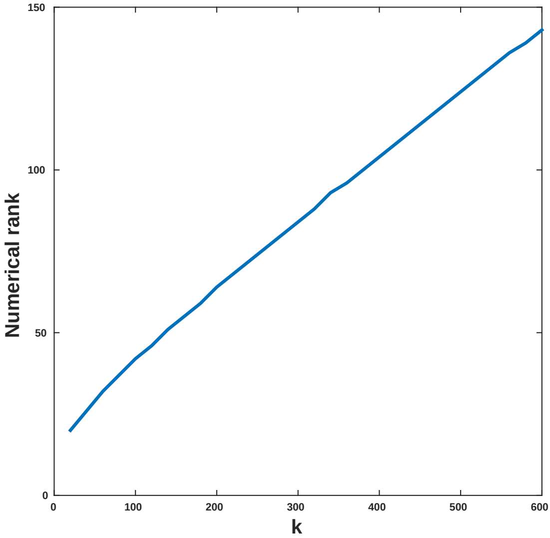
Interaction between separated regions X, Y:





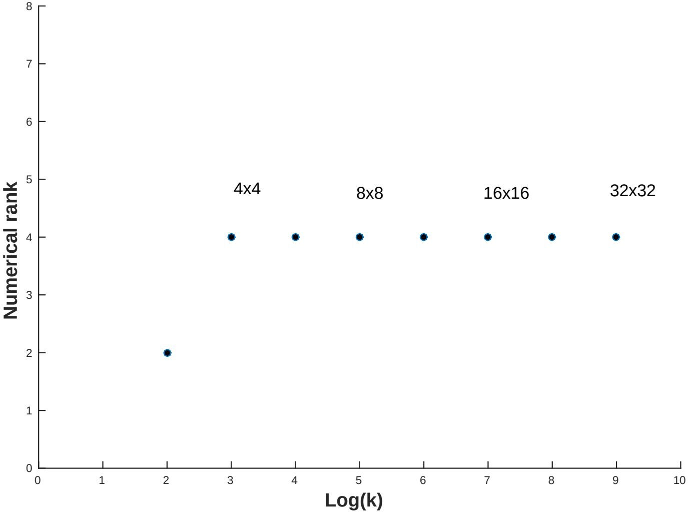
Helmholtz single-layer potential

Numerical rank of $\mathcal{S}_{X \times Y}$ as a function of k:



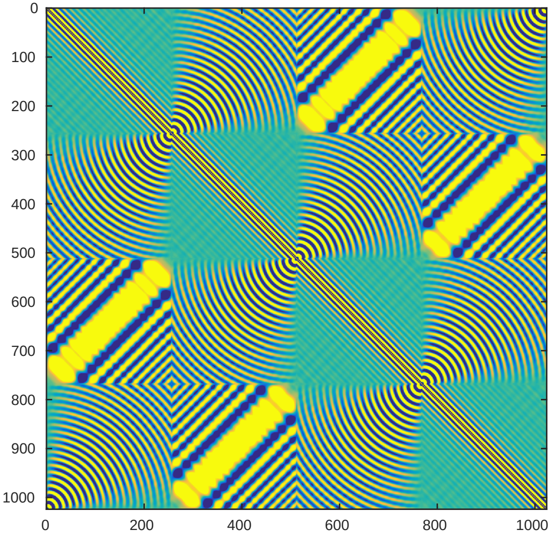
Helmholtz single-layer potential

Ranks of all subblocks with $\rho(X)\rho(Y) = O(1/k)$:



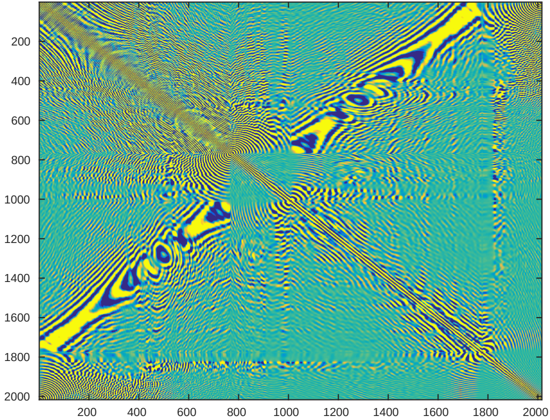
Impedance operators

Constant coefficient on square domain, more complicated structure:



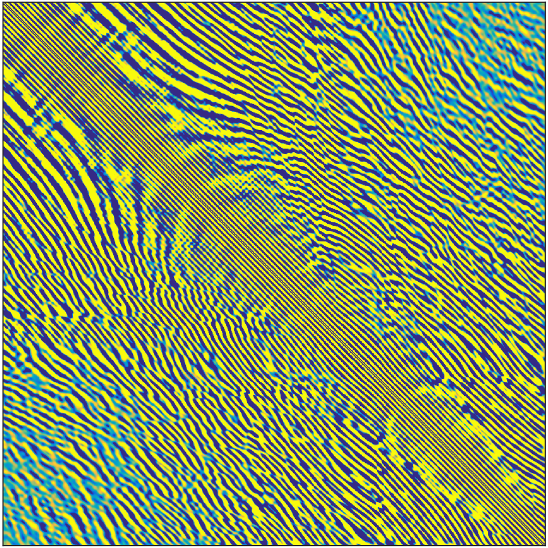
Impedance operators

Marmousi model (variable coefficient), even more complicated:



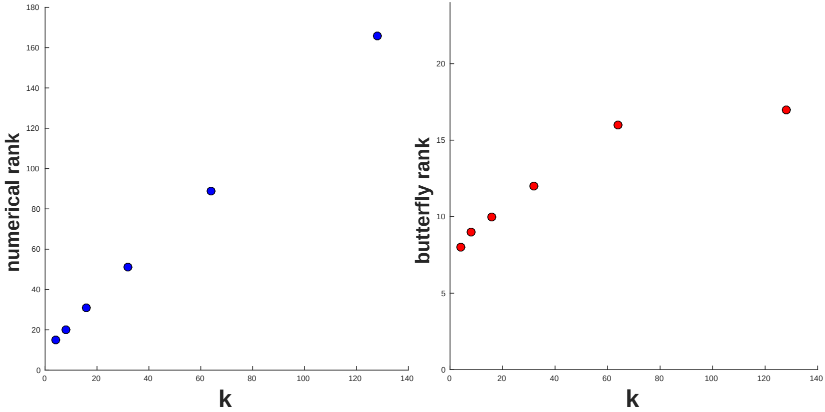
Impedance operators

Upper left block:



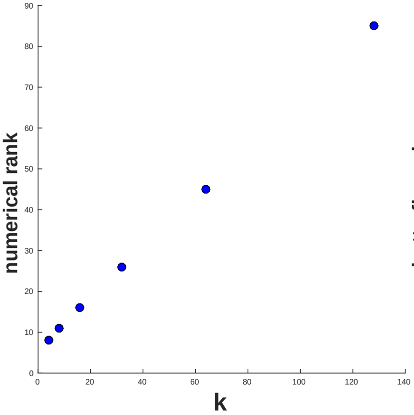
Impedance butterfly compressibility

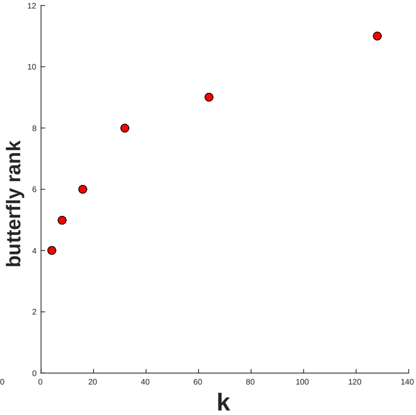
Constant coefficient operator, off-diagonal blocks:



Impedance butterfly compressibility

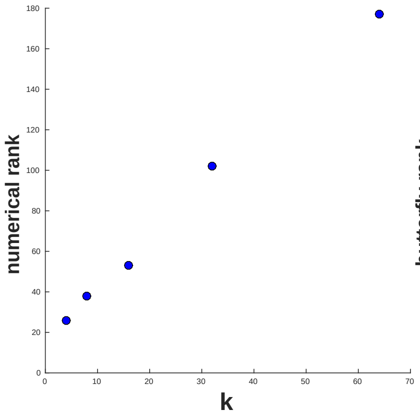
Constant coefficient operator, separated blocks:

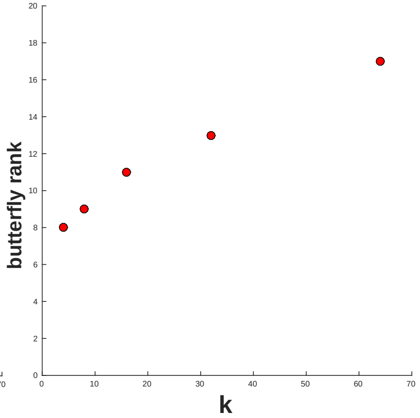




Impedance butterfly compressibility

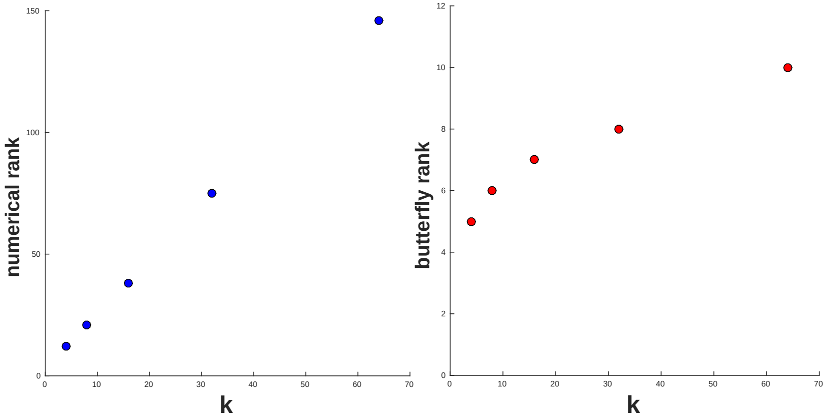
Variable coefficient operator, off-diagonal blocks:





Impedance butterfly compressibility

Variable coefficient operator, separated blocks:



Outlook

- Multiscale compressibility experimentally appears possible.
- Factorization/inversion is a big open question.
- Michielssen et al. (2013 - present): LU-based solver for Helmholtz kernels. Can it be extended?
- ULV-type factorizations for multi-level butterfly?
- Alternatives to butterfly?