Qualifying Exam Transcript

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Remark. This transcript is only a rough recollection of the proceedings. None of the below is verbatim.

Wilkening: What topic do you want to start with?
Wong: Let's do it in order: PDE, numerical linear algebra, functional analysis.
Zworski: What is the Legendre transform?
Wong: Given a map $L : \mathbb{R}^n \to \mathbb{R}$ convex and superlinear, I define $L^*(p) = \sup_v \{p \cdot v - L(v)\}$.
Zworski: What do you mean by superlinear?
Wong: I mean that $\lim_{\|v\| \to \infty} |L(v)|/\|v\| \to \infty$.
Wilkening: Are you sure you want absolute values on the top? Can $L(v)$ go to negative infinity?
Wong: (*I think about this for a moment*) I'm not sure...
Zworski/Wilkening: Why do you need L to be convex for this supremum to be defined?
Wong: I guess it doesn't need to be convex, but I want the Legendre transform to be an involution.
Wilkening: So what condition do you want on L if it's not convex?
Wong: I guess I need to remove the absolute values, so $\lim_{\|v\| \to \infty} L(v)/\|v\| \to +\infty$.
Strain: If L is differentiable, how do you get L^*?
Wong: I can just differentiate and set to zero to calculate: $0 = p - \partial L/\partial v$.
Govindjee: When L is convex there are much nicer properties, but the case when L is not convex is important and useful. Can you give me a function that is superlinear but not convex?
Wong: (*I draw a quartic polynomial with two wells*)
Govindjee: If you take the Legendre transform twice, what do you get? What geometric properties does the result have?
Wong: I don't know, but I would conjecture that the result is convex.
Govindjee: Your conjecture is correct.
Wong: Ah, so then I'm guessing that the epigraph of the result is some convex set related to the epigraph of the original function.
Govindjee: So the result will be equal to the original function except that when L is not convex, it'll sweep across.
Zworski: So what's the connection to PDE?
Wong: If I have a vector function H and $u_t + H(Du) = 0$, $u(0) = g$, then I can recast this as a minimization problem of the form

$$u(x,t) = \inf_{w(t)=x} \left\{ \int_0^t L(Dw)\, ds + g(w(0)) \right\},$$

where the term at the end is cooked in to account for the boundary conditions, and $H^* = L$. In physics, you can think of $u(x,t)$ as the value of the action, and you're searching over paths $w(\cdot)$ to

find the one of least resistance.
Zworski: Why can't H also be a function of x?
Wong: Oh...I guess it can be a function of x, since the Legendre transform doesn't see x. (*I modify my minimization problem*)
Zworski: And how do I solve this PDE?
Wong: I can use the method of characteristics. I have

$$\dot{x} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial x.$$

There's also third characteristic equation but it doesn't figure into it.
Zworski: How does this relate to L?
Wong: Hm...(*I think for a bit, as I am not sure*) Oh, if you use a change of variables to change the above equations in terms of L and v, you get the Euler-Lagrange equations.
Strain: Let's talk about linear PDE.
Wilkening: What is the Hille-Yosida theorem?
Wong: Do you want me to just state the theorem, or give some definitions first?
Wilkening: Just state the theorem.
Wong: If $A : D(A) \to X$ is a possibly unbounded, densely defined linear map on a Banach space X, then we want to know if there is a strongly continuous semigroup of operators in X whose infinitesimal generator is A. The Hille-Yosida theorem says that is true if there is some $\gamma \in \mathbb{R}$ such that if $\lambda > \gamma$ implies $\lambda \in \rho(A)$, the resolvent set, and if the resolvent map has the norm inequality

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \gamma)^n}.$$

Wilkening: Let's apply this to a PDE. Somebody once told me a PDE that arises in physics, that has the form, for x on the positive real line,

$$\frac{\partial f}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^2 e^{-x^2}\frac{\partial}{\partial x}\left[f e^{x^2}\right]\right].$$

Can you try to apply Hille-Yosida to that?
Wong: Uh oh...I guess I can try. (*I begin to expand the equation using product rule*)
Govindjee: You don't want to do that, because you're going to lose the self-adjoint structure. Do you know the form of Sturm-Liouville operators in ODE theory?
Wong: Yes, I think so...(*I start to write down the form of a Sturm-Liouville operator*)
Wilkening: Try the substitution $u = fe^{x^2}$.
Wong: Okay, so then I get $e^{-x^2}u_t = x^2\partial_x(x^2 e^{-x^2}u_x)$. (*slightly unsure*) This looks familiar to me from physics...
Strain: So what is the weak formulation of this? You want to multiply by a test function...
Wong: Oh, so I guess I get $\langle x^2 e^{-x^2}u_t, \phi\rangle_{L^2} = -\langle x^2 e^{-x^2}u_x, \phi_x\rangle_{L^2}$ using integration by parts?
Govindjee: I think you want to make it clear what the weight is here.
Wong: (*After some puzzlement*) Ok, so I can just write this using the measure $d\mu = x^2 e^{-x^2}dx$, so then it looks like $\langle u_t, \phi\rangle_\mu = -\langle u_x, \phi_x\rangle_\mu$.
Govindjee/Wilkening: What about the boundary term?
Wong: Oh, so I guess that would be $[x^2 e^{-x^2}u_x\phi]_0^\infty$. So I guess I want u to be Schwartz class? Or maybe it just grows subexponentially?
Wilkening: It can even grow exponentially, right? So we can pick our space to be a nice function space.

Wong: Yeah, I guess. Ok, so I need to look at the spectrum. The operator is self-adjoint, where $u_t = Au$.
Zworski: It's not entirely clear yet that the operator is self-adjoint.
Wong: Well, it's at least symmetric.
Zworski: Yes, but that's not enough. Then you have to use a Friedrichs extension to show that you can extend to a self-adjoint operator on the appropriate space.
Wong: Ah, I guess I don't know anything about that. I guess I remember that during the colloquium a few months ago the speaker mentioning about how a lot of issues in physics arise from operators being symmetric but not self-adjoint.
Zworski: We can just assume it's self-adjoint.
Govindjee: So what is the spectrum of a self-adjoint operator? Can it be anywhere on the complex plane?
Wong: It has to be on the real line. But I need to show that the spectrum is bounded above.
Govindjee: Can you show that it can't have positive eigenvalues at all? What can I plug in for ϕ so that I get negative something...
Wong: So if I compute the numerical range $\langle Au, u\rangle_\mu = -\int u_x^2 \, d\mu \le 0$, then this shows that the operator has no positive eigenvalues. (*feeling unsure*) But how do I show there can't be elements of the spectrum at all on the positive real line?
Zworski: It's self-adjoint, you can use the spectral theorem.
Wong: So that says that you can write A as some integral over the spectrum of A with a projection-valued measure. So that means...? I also need to show that the resolvent is bounded (*they assure me it'll work*)
Govindjee: You can get the bound for the resolvent pretty quickly.
Wong: So then that case, we get that the solution is given by $u(x,t) = S(t)u_0(x)$, where $S(t)$ is the semigroup guaranteed to exist by the theorem.
Zworski: Isn't the Hille-Yosida an "if and only if" statement? What is the other direction?
Wong: Oh, so if I have a semigroup, then the theorem says that there exists a unique infinitesimal generator, and...
Zworski: What bound do I need on the operators?
Wong: Oh right, so I need that there exists M, γ such that $\|S(t)\| \le Me^{\gamma t}$.
Zworski: Yes, because otherwise I can construct semigroups that grow very fast and there won't be a generator.
Wilkening: Are you sure you can construct it?
Wong: I guess that makes sense, because we need this exponential growth condition, because to prove that A exists, we basically construct the resolvent by using the Laplace transform on $S(t)$, so that integral only converges if there is an exponential bound.
Strain: So how about the heat equation?
Wong: Well, then we can apply Hille-Yosida on the operator $A = -\Delta$.
Strain: Let's not talk about semigroups anymore. What is the fundamental solution in free space?
Wong: Well, it's $\Phi(x,t) = C_n t^{-n/2}\exp(-|x|^2/(4t))$, but I forget what the constant is. You can obtain this by using the Fourier transform.
Strain: Ok, so how do you get the fundamental solution for the heat equation with periodic boundary conditions? Say it's just on the real line.
Wong: We can use the method of images, so then we get

$$\sum_{n\in\mathbb{Z}} C_n t^{-n/2}\exp(-|x-n|^2/(4t)).$$

I don't know if I have to change the constant though...

Strain: When you're near one singularity, does the value of the function depend on the other terms?
Wong: I guess not, so the constant is the same.
Strain: Okay, so what if try to find it a different way. What if we use the Fourier transform? What is the equation in Fourier space?
Wong: (*feeling unsure*) Okay, so this using Fourier series then? Hmm, so I have $\hat{u}_t + |k|^2\hat{u} = 0, \hat{u}(0) = \hat{u}_0$. So we have

$$u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ikx}.$$

Strain: The function on the left depends on t. Where's the t on the right side?
Wong: Oh, right, so it should be $\hat{u}(k,t)$.
Strain: So how do we get a fundamental solution?
Wong: (*I draw a blank for a bit*) So we want u_0 to be the delta function.
Strain: What's the Fourier transform of the delta function?
Wong: I should really know this, but I don't remember. So I guess I should work it out. (*I start to write down the convolution identity*) So the Fourier coefficients of the δ function are 1 at a point, and 0 elsewhere...?
Strain: No, that would be e^{ikx}. What do you get when you transform the convolution?
Wong: (*I'm drawing a blank for a while, but finally I'm led to write*) Oh, so $\hat{\delta}\hat{f} = \hat{f}$, so $\hat{\delta} \equiv 1$. Yeah, I should've known that, sorry.
Strain: Ok, so what is $\hat{u}(k,t)$?
Wong: Ah... (*I finally figured out what he was asking*), so then $\hat{u}(k,t) = \hat{u}_0 e^{-|k|^2 t} = e^{-|k|^2 t}$ for each k, so then the fundamental solution is

$$\Phi(x,t) = \sum_{k\in\mathbb{Z}} e^{-|k|^2 t}e^{ikx}.$$

Strain: So you have something like the Poisson summation formula. Compare your answer here to the one you got using the method of images. Is it obvious that they're equal?
Wong: Well, if you take the Fourier transform of one...
Strain: Think about what you're saying. Both functions are in position space already.
Wong: Oh, I guess then it's not obvious to me.
Zworski: It was not originally obvious.
Govindjee: Isn't this just some identity...?
Strain: One man's identity is another man's theorem. (*Or some joke like that*)
Wilkening: Okay, let's move on to numerical linear algebra.
Strain: Prove that Gaussian elimination is stable! (*smiles*)
Zworski: (*turning to Strain*) Is it stable?
Strain: Nobody knows!
Govindjee: Tell me about Krylov subspace methods for computing eigenvalues.
Wong: So we're given $A \in \mathbb{R}^{m\times m}$, not necessarily symmetric, and suppose $A = QHQ^*$ is its Hessenberg form.
Zworski: What are Q and H?
Wong: Oh, so Q is unitary and H is a matrix with only zeros below the first subdiagonal. So the idea is that I can truncate this matrix by using the identity $AQ_n = Q_nH_n$, where Q_n is the first n columns of Q, and H_n is the truncated section of H. We find Q_n by computing an orthogonal basis for the subspace span$\{b, Ab, \ldots, A^{n-1}b\}$, essentially just using Gram-Schmidt, using b a random

vector.
Zworski: b is random?
Wong: Yeah. Well, we essentially want to take a typical b such that b isn't already contained in...
Wilkening: Contained in an eigenspace.
Wong: Yeah.
Zworski: What is n?
Wong: n is hopefully much smaller than m. So the idea is that we want to find this H_n, and then approximate the first n eigenvalues of A by the eigenvalues of H_n.
Strain: So, why do we need to do this? Why can't I just directly compute all the eigenvalues?
Wong: Well, it is expensive.
Strain: Can I compute the eigenvalues exactly? For what m can't I do this? (*somewhat jokingly*) What French mathematician says you can't?
Wong: Abel? He isn't French! But yeah, you can't do it for $m > 4$ because you can't solve the quintic.
Strain: Galois!
Wong: Oh, I thought Abel was the guy who originally proved you couldn't do the solvability stuff for certain field extensions.
Wilkening: Wait, so I think there's something wrong here. I don't think H_n can be the $n \times n$ section of H.
Strain: Yeah, if it were true that $AQ_n = Q_n H_n$, then Q_n would form an invariant subspace.
Wong: Oh, right, we need to shift up by one. So there should be an $n + 1$ somewhere...
Wilkening: So H_n should be the $(n + 1) \times n$ section of H.
Wong: Yeah, so it should be $AQ_n = Q_{n+1} H_n$.
Govindjee: And then we chop it down to $n \times n$ to calculate the eigenvalues. So, how do the eigenvalues of the truncation converge to the real eigenvalues?
Wong: Well, experimentally, the eigenvalues at the extremes converge first. But I've only learned that from reading about people observing that experimentally. I actually don't know, is there something guaranteeing this?
Wilkening: I think that one can construct an A such that the procedure fails arbitrarily badly unless you compute the whole vector space.
Govindjee: Is there any problem with this method if n is large?
Wong: Yes, if n is large, then my vectors can start failing to be orthogonal, when doing Gram-Schmidt. So one thing I can do is orthogonalize again.
Govindjee: Basically you want to restart the process.
Wong: Ah, yeah. There can also be an issue when A is symmetric, in which case H is tridiagonal, because then you calculate the columns of Q_n using a three-term recurrence.
Wilkening: So you're worried that after a while q_n might not be orthogonal to q_1.
Wong: Right, so at some point I should just manually reorthogonalize.
Strain: Why is it bad that they're not orthogonal?
Wong: Well, then I can start to accumulate errors. I don't want the inner product of any q_j's to greatly exceed the machine precision.
Govindjee: So, what happens if I shift A by a constant; in other words, if I do this procedure to $A - \mu I$.
Wong: (*I think for a moment*) Nothing.
Govindjee: Right, so this method is shift invariant.
Strain: How could I use this property to try to more quickly get some particular eigenvalue? What shift would I use?
Wong: You could shift by that eigenvalue, or some approximation to it.

Strain: Right.
Wilkening: Let's talk about stability theory. Can you define backwards stability for me?
Wong: So if $x \mapsto f(x)$ is some map, and an algorithm computes $\tilde{f}(x)$, then that algorithm is backwards stable if there exists some $\tilde{x}$ such that the exact equality $\tilde{f}(x) = f(\tilde{x})$ holds.
Strain: So, what kinds of maps can have algorithms that are backwards stable, morally speaking?
Zworski: (*to Strain*) Morally speaking – what does that mean?
Wong: I'm not sure.
Strain: If I have a vector-valued map f from a low dimensional vector space to a higher dimensional vector space, can errors from the high dimensional space be thrown onto the low dimensional vector space?
Wong: I'm not sure why that would affect anything. The error doesn't have to be unique.
Wilkening: Try this for the tensor product of two vectors. Look at it componentwise.
Wong: Okay, so then xy^* has components $x_i y_j$, so then an algorithm which computes it will have components $x_i y_j(1 + \delta_{ij})$ for some floating point error δ_{ij}.
Strain: What are the δ_{ij}? Are they just anything?
Wong: They're controlled by a constant times the machine precision.
Strain: The constant is equal to 1, because we're doing exactly one multiplication.
Wong: Oh, right! These are just real numbers. So there no way to attribute the forward error $x_i y_j(1 + \delta_{ij})$ unless the δ_{ij}'s are all equal, because we can't throw different errors onto the same x and y.
Strain: What are the dimensions of the map?
Wong: So the output is n^2 dimension and the input is n-dimensional.
Wilkening: $2n$-dimensional. So, how about we try to show the stability of some algorithm...
Strain: Can you show that iterative refinement in mixed precision has error which converges to machine precision? I'm just kidding.
Wilkening: Can you show that Givens rotations are backward stable?
Wong: I actually don't know anything about Givens rotations.
Wilkening: Okay, how about the eigenvalue problem. Maybe we can do a forward error estimate? Let A be symmetric and $A = U\Lambda U^{-1}$.
Strain: What is the eigenvalue solver? I normally think of backward stability as a property attributed to an algorithm.
Wilkening: Okay, so just assume that there is backward stability. Can you derive an error estimate for the eigenvalues?
Wong: I can compute something componentwise. By backward stability, we have

$$(\lambda + \delta\lambda)(u + \delta u) = (A + \delta A)(u + \delta u).$$

Then, matching the first order terms, we get

$$\lambda\delta u + \delta\lambda u = A\delta u + \delta A u + O(\epsilon^2).$$

A is symmetric, so if we hit this by u^* on the left, and cancel one of the terms, we get

$$\delta\lambda = \frac{u^*\delta A u}{\|u\|^2}.$$

Zworski: This comes from the spectral theorem.
Wilkening: Can you derive some sort of estimate using the matrix U?
Wong: Oh, so I guess there is a stronger result attributed to Weyl...

Wilkening: Yes, that is what I was trying to get at.
Wong: Well, I think we have each eigenvalue obeys the perturbation estimate $|\delta\lambda| \leq \kappa_2(U)\|\delta A\|_2$, where A doesn't even need to be symmetric.
Wilkening: Are you sure it isn't supposed to be the condition number of A?
Wong: I don't think so. And when U is unitary, then the condition number is 1, so we have $|\delta\lambda| \leq \|\delta A\|_2$.
Zworski: This still basically comes from the spectral theorem.
Strain: Do you know what a left eigenvector is?
Wong: Yes.
Strain: So if A were not symmetric, how could you modify your first perturbation?
Wong: So if v is the left eigenvector, then my new estimate is $|\delta\lambda| \leq v^*\delta A u/(v^*u)$.
Strain: So if u, v are unit vectors, what is their inner product?
Wong: The angle between them.
Strain: You mean the cosine of their angle.
Wong: Oh, right. So then the estimate is controlled by $\sec\theta$, and this blows up when the left and right eigenvectors are orthogonal. This happens when the eigenspace is degenerate, so A is not even diagonalizable.
Strain: And what about your second estimate?
Wong: So when A is very far from being orthogonally diagonalizable, then the error estimate is really bad, because it is controlled by $\kappa_2(U)$, which may be large. I guess I should use a different letter than U when U isn't even unitary.
Strain: What about if A is anti-symmetric?
Wong: You mean if $A^T = -A$? But then A is normal.
Strain: Oh, right.
Wilkening: Let's move on to functional analysis.
Strain: I think Maciej has a question he wants to ask.
Zworski: So this combines some functional analysis with some of the stuff from PDE. Let's consider the PDE $\operatorname{div}(\gamma Du) = 0$ on Ω, $u = f$ on $\partial\Omega$. Using the variational formulation, can you show that there must exist a weak solution $u \in H^1(\Omega)$ given that $f \in H^{1/2}(\partial\Omega)$?
Wilkening: γ is a function in space?
Zworski: Of course, otherwise it would just be a constant.
Wong: So I guess I should probably work backwards from the Euler-Lagrange equations. Well, to start, this is going to be the infimum over something...
Govindjee: That's always a good place to start.
Wong: It's going to be an infimum of the quantity $\int_\Omega L(Dv)\,dx$ over the set of functions in $v \in H^1(\Omega)$ whose image in the trace operator Tv is equal to $f \in H^{1/2}(\Omega)$. But, what is the Lagrangian...
Govindjee: Maybe you should write the weak formulation.
Wong: Okay, so the weak formulation is, for $\phi \in C_c^\infty(\Omega)$,

$$-\int_\Omega \gamma Du D\phi\,dx = 0.$$

(*I get stuck for a moment*) So the Lagrangian...
Govindjee: Let's start with a one half out front...
Wong: Oh, so the Lagrangian should be $L = \frac{1}{2}\gamma|Dv|^2$.
Zworski: Okay, so how do I guarantee that the minimizer exists? What's the magic word?
Wong: I'm not sure...well, you need the Hessian of L to be positive. So γ needs to be a positive function.

Zworski: Okay, that gives you convexity –
Strain/Govindjee: (*amused*) Bad boy! You gave away the magic word!
Zworski: No, because convexity only gives you uniqueness of the minimizer, right? There's another magic word.
Strain: (*excited*) Oh, is it... (*whispers in Zworski's ear*)
Wong: Right, but I cannot seem to remember the condition for existence.
Govindjee: Who did you take calculus of variations from?
Wong: Professor Evans.
Govindjee: He definitely should've mentioned this.
Wong: (*I think for a minute or two but I don't get anywhere*) Yeah, I just don't remember.
Zworski: You need the Lagrangian to be coercive for weak minimizers to exist.
Wong: Oh...I'm sorry, I must admit I had forgotten that.
Zworski: Okay, so there is one last condition you need. It might seem trivial, but it's still important.
Wong: Oh, well, we need the action functional not to be infinity...
Zworski: Well, yes, but not that. How do we even know the infimum is defined?
Wong: Ah, you need that the set over which the infimum is taken to be nonempty.
Zworski: Right, and how do we know this?
Wong: It is known that the trace operator $T : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is surjective.
Zworski: How do you show this?
Wong: Given $f \in H^{1/2}(\partial\Omega)$, I can locally extend f to an H^1 function in a neighborhood of the boundary.
Zworski: Right. So suppose that f gives rise to the extension u which is the weak solution to the PDE. What is a typical sort of inequality do we have related to this extension?
Wong: Well, I guess there exists $C > 0$ such that $\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}$.
Zworski: So what does that imply about the map?
Wong: It's continuous.
Zworski: Okay, so suppose you didn't know this result, but we have the existence of weak solutions for the PDE by using the existence of weak minimizers for the variational problem. How could we prove this map is continuous?
Wong: Hmmm...
Govindjee: Write down the map. What properties does it have?
Wong: So we can define the map $E : H^{1/2}(\partial\Omega) \to H^1(\Omega)$, and it's linear, because I guess if you add the traces then the new minimizer is the sum of the original minimizers.
Govindjee: You could just use the linearity in the original PDE.
Zworski: So how can you prove that this linear map is continuous? You should use one of the items on your syllabus. I'm looking at it right now. Do you want to see your syllabus?
Wong: I want to try to figure this out.
Govindjee: It's like a matter of pride!
Wong: (*very unsure, I think for a minute*) Maybe I could use the closed graph theorem?
Zworski: Yes!
Wong: (*still unsure*) Okay, hm, so, if it has a closed graph, then it is a bounded operator. I don't know how to show it has a closed graph. Okay, well, so this means that I have a sequence $(f_n, Ef_n) \to (f, g)$ and I want to show that $g = Ef$.
Zworski: What does that convergence mean?
Wong: It is with respect to the product topology from the respective norms of the two spaces.
Govindjee: So how can we show that $g = Ef$? You want to show that g is a weak solution. Write down the weak formulation.

Wong: So we have that, for ϕ a test function,

$$\int_\Omega -\gamma D(Ef_n - g) \cdot D\phi \, dx = 0.$$

Strain: How can you control the size of the dot product?
Wong: Oh, I can use the Schwarz inequality. Okay, so then

$$\left| \int_\Omega -\gamma D(Ef_n - g) \cdot D\phi \, dx \right| \leq C \|Ef_n - g\|_{H^1(\Omega)} \|D\phi\|_{L^2} \to 0.$$

(*sudden realization*) Oh, so that means that the original quantity vanishes, so then g is a weak solution, and its trace is by assumption f. Okay, so the closed graph theorem then says that the map is continuous.
Govindjee: So, I have a question, for my own education. How do you define the space $H^{1/2}(\partial\Omega)$?
Wong: Okay, so assuming the boundary is regular, I want to define H^s on the flat space...
Zworski/Wilkening: Are you sure you want to do that?
Wong: Yeah, I basically just want to parametrize $\partial\Omega$ and define H^s on the usual Lebesgue measure, then pass to the boundary measure. So, for any s, I can define $H^s(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) : \left((1+|k|^2)^{s/2}\hat{u}\right)^\vee \in L^2\}$.
Strain: Do you need the inverse Fourier transform?
Wong: Oh, no, I don't.
Wilkening: Are you sure that u can be any distribution? Not something in the dual of the Schwartz class?
Wong: Oh, yeah, you're probably right. So it should be $u \in \mathcal{S}'(\mathbb{R}^n)$. Okay, so then, for any Ω, we can define $H^s(\Omega)$ by taking the restriction...
Zworski: Wait a minute. He's asking about defining H^s on some submanifold rather than on a domain. So maybe we should define $H^s(\partial\Omega)$ by defining its norm.
Wong: (*A little confused. I think for a bit*) Okay, so maybe I should define $\|f\|_{H^{1/2}(\partial\Omega)}$ by taking the supremum...
Wilkening: No, you don't want the supremum! Because a function could have trace f but then be huge on the interior.
Wong: Oh yeah, oops. Okay, so I should be using the infimum, so I define

$$\|f\|_{H^{1/2}(\partial\Omega)} := \inf_{u \in H^1(\Omega): Tu = f} \|u\|_{H^1(\Omega)}.$$

Does there need to be a constant? What would the constant be?
Zworski: It doesn't matter! We just want to define the space. If you define it by unravelling the boundary into flat space as you suggested before, then you'd have all sorts of complicated constants, but it doesn't matter.
Strain/Govindjee: This is an interesting definition but somehow not satisfying.
Zworski: Okay, well, how about $H^{1/2}(S^1)$. How would you define that? Maybe you want to use the Fourier series definition that you gave above?
Wong: Oh, so maybe I want the $H^{1/2}(S^1)$ norm to be defined by

$$\|u\|_{H^{1/2}(S^1)} = \sum_{k \in \mathbb{Z}} \hat{u}(k)|k|^{1/2}.$$

Zworski: Watch out, I'm concerned about when $k = 0$...

Wong: Oops, so I should change it to

$$\|u\|_{H^{1/2}(S^1)} = \sum_{k\in\mathbb{Z}} \hat{u}(k)(1+|k|^2)^{1/4}.$$

Strain: That's much more satisfying. So I think I'm satisfied for today. (*the others agree*) I guess we're done now.